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Complementary finite-element method for finite deformation nonsmooth mechanics

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Abstract. The complementary finite-element programming method and algorithm for solving finite deformation problems in nonsmooth mechanics are presented. This method provides a dual approach for the numerical solutions of the mixed boundary-value problem governed by nonsmooth physical laws. Application to non-smooth plastic flow is illustrated.

1. Introduction

Numerical solution of nonlinear mechanics, especially for geometrical nonlinear problems governed by the nonsmooth physical laws, has always presented serious difficulties for applied mathematicians and engineers. The objective of this paper is to study the complementary variational approach and finite-element method for finite deformation nonsmooth mechanics. Such a research is of particular interest, since the finite deformation systems governed by nonsmooth physical laws occur in many fields of applications. See, for example, P. D. Panagiotopoulos [1], J. E. Taylor [2] for variety of problems arising in Mechanics and Engineering.

It is known that in the early years of finite-element analysis, there were two basic methods: the 'displacement method', which is based on the potential energy principle, and the 'force method', which is based on the complementary energy principle. In linear elasticity analysis, the construction of the algebraic equations for large-scale structures, via the force method, is awkward and inefficient in comparison with the displacement method process. So the vast majority of practitioners today regard finite element structural analysis as a method based on assumed displacements. In finite deformation nonsmooth mechanics, the situation is different. Since the governing equation is nonlinear and the potential energy functional is not sure convex, the displacement method sometimes is very difficult, even impossible. However, on the opposite side, the dual approach may provide a potentially useful method for solving nonsmooth systems. Perhaps this is what is the so-called complementarity.

Complementary variational principles and methods in solid mechanics have been the objects of fruitful scientific preoccupation of many a distinguished mechanician such as Reissner [3], Fraeijis de Veubeke [4], Oden and Reddy [5], Pian and Tong [6] and many more. During the past 20 years the complementary duality theory for geometrical linear systems, i.e. infinitesimal deformation systems, has been extended and generalized in various directions to study a wide class of problems arising in optimization and control, mechanics, operations research, fluid dynamics, economics and transportation equilibrium. See, for example, A. M. Arthurs [7], I. Ekeland and R. Temam [8], M. J. Sewell [9], etc.

For geometrical nonlinear systems, i.e. large deformation problems, the duality theory has been studied by Gao and Strang [10]. By introducing a so-called complementary gap function, the complementary dual variational principles in finite deformation theory has been established. Applications to nonsmooth mechanics show that this gap plays a key role in the analysis of geometrical nonlinear mechanics (see Gao *et al.* [11–16].

The title of a recent paper by Felippa [17] asked the question: "Will the force method come back?" Professor Gallagher [18] said: "I prefer to ask, however, can we expect to be able to employ, routinely, a method that is 'dual', or opposite-hand, to the displacement/stiffness method?"

Based on the general duality theory established in [10], the purpose of the present paper is to offer a complementary variational approach to finite-element analysis of more general problems in finite deformation nonsmooth mechanics. Compared with the traditional displacement, or Ritz method, this method has the following advantages:

(1) Provides a dual approach to the primal problem. Generally speaking, the displacement method gives the upper bounds for the nonlinear problem, however, the dual method will give the lower bound. This is the meaning of the complementarity. This advantage is important for problems where lower-bound solutions are desired. For instance, in plastic limit analysis, the engineer wants to know the lower-bound of the safety loading factor for structural design. So the complementary finite-element method provides a direct way to solve these very important problems.

(2) Reduces the nonconvex primal problem to a convex dual problem. In nonsmooth systems, the total potential energy functional could be nonconvex. But by using the Legendre-Fenchel transformation, the conjugate functional is always convex. So we can use the convex programming method to solve the nonsmooth system.

(3) Reduces the degree-of-freedom in nonlinear programming. For most physical nonlinear problems, say the plastic plane flow, suppose that the degrees-of-freedom for the discrete primal problem is $2N$, for dual problem it is $3N$, then by the method we proposed, we have only N degrees-of-freedom. This advantage is important for large-scale nonlinear system computing.

(4) Reduces the weak nonlinear problem to a coupled quadratic problem. In physical linear systems (for example, the large displacement but small strain deformation), the total potential is a quadratic function of the strain tensor. But the primal variational problem is still a nonlinear problem due to the geometrical nonlinearity. In the dual problem we use stress, the dual variable of strain as the variational argument and the total complementary energy is quadratic in both stress and displacement, (see Gao and Cheung [14], Yau and Gao [15]). Then, by the dual approach, we can suggest a linear iteration algorithm for solving a nonlinear boundary value problem.

(5) Reduces the order of differentiation. In nonlinear structural analysis, such as nonlinear elastic plates, shells, etc., the unknown functions in the primal problem should be at least twice differentiable. By using the traditional Ritz finite-element method, we need at least second-order interpolation. But in the dual problem, the trivial function needs to be in C^1 or C^0 only. So we can use the linear or constant interpolation in finite-element approach.

It is because of these advantages that the complementary finite-element method may provide an effective approach for numerical solutions of nonlinear, nonsmooth systems.

2. General problems and complementary variational approach

In order to describe the finite deformation nonsmooth mechanics, the operator notations and the theory of convex analysis have to be used in this paper; such notations and theory have been extensively used in [8, 10]. Let us consider the numerical method for the most general system of the nonsmooth variational problem:

PROBLEM 1. *Let $\mathcal{U}, \mathcal{E}$ be the generalized displacement space and strain space, respectively, $\Lambda : \mathcal{U} \to \mathcal{E}$ the finite deformation operator. Suppose that the total potential $J(v, \Lambda v) : \mathcal{U} \to \mathbf{R}$ is a lower semi-continuous functional. Find $u \in \mathcal{U}$ such that*

$$J(u, \Lambda u) = \inf_{v \in \mathcal{U}} J(v, \Lambda v). \tag{1}$$

In many cases, the nonsmooth functional J can be written as

$$J(v, \epsilon) = W(\epsilon) + F(v),$$

where $W : \mathcal{U} \to \mathbf{R}$ and $F : \mathcal{U} \to \mathbf{R}$ are nonsmooth functionals. For a mathematical-physics system, the function W denotes the internal energy of the system. Its subgradient ∂W is a convex subset of the general stress space $\mathcal{E}^*$, the dual space of $\mathcal{E}$:

$$\partial W(\epsilon) := \{\sigma \in \mathcal{E}^* | W(e) - W(\epsilon) \geqslant \langle \sigma, e - \epsilon \rangle, \quad \forall e \in \mathcal{E}\}.$$

If W is a smooth function, $\partial W(\epsilon)$ has only one element, i.e. $\sigma = \partial W(\epsilon) = \partial W / \partial \epsilon$. The physical meaning of this subdifferential constitutive relation is shown in a later section of this paper. Meanwhile, the function F denotes the external energy of the system. Its subgradient ∂F controls the physical relation between the configuration variable space $\mathcal{U}$ and the source variable space $\mathcal{U}^*$.

If $\Lambda : \mathcal{U} \to \mathcal{E}$ is a linear map, then we call the system a *geometrical linear* system. Otherwise, the system is *geometrical nonlinear*.

For a geometrical linear system, the Euler–Lagrangian for problem 1 is the following subdifferential inclusion

$$0 \in \Lambda^* \partial W(\Lambda u) + \partial F(u). \tag{2}$$

Here $\Lambda^* : \mathcal{E}^* \to \mathcal{U}^*$ is the adjoint operator of Λ defined by the following Gauss-Green theorem:

$$\langle \Lambda u, \sigma \rangle = \langle u, \Lambda^* \sigma \rangle.$$

By introducing a pair of intermediate physical variables $\epsilon \in \mathcal{E}$ and its dual $\sigma \in \mathcal{E}^*$, this subdifferential inclusion can be split into three equations:

(1) Geometric Eqn. $\quad \epsilon = \Lambda u, \qquad (3)$

(2) Physical Eqns. $\quad \sigma \in \partial W(\epsilon) \text{ and } -t \in \partial F(u), \qquad (4)$

(3) Equilibrium Eqn. $\quad \Lambda^* \sigma = t. \qquad (5)$

The physical equation $\sigma \in \partial W(\epsilon)$ controls the internal property of the system, which is usually called the constitutive equation. However, $-t \in \partial F(u)$ controls the external property of the system and gives the boundary conditions. We can see a nice symmetry in geometrical linear systems.

However, for geometrical nonlinear system, where $\Lambda : \mathcal{U} \to \mathcal{E}$ is nonlinear, such a symmetry is broken. The geometrical and physical equations in this system are the same as (3) and (4), respectively, but the equilibrium equation should be (see [10]):

$$\Lambda_t^* \sigma = t. \tag{6}$$

Here Λ_t^* is the adjoint operator of Λ_t, the Gâteaux-derivative of Λ, which depends on the variable u. So the Euler-Lagrange inclusion can be given by

$$0 \in \Lambda_t^* \partial W(\Lambda u) + \partial F(u). \tag{7}$$

This general subdifferential equation governs many finite-deformation nonsmooth systems (see [11–16]).

Example: Let us consider the mixed boundary-value problem for a 3-dimensional nonlinear elastic system. The domain $\Omega \in \mathbf{R}^3$ is an open, connected, bounded subset of $\mathbf{R}^3$ with piecewise Lipschitz boundary $\Gamma = \partial\Omega = \Gamma_u \cup \Gamma_t$. On the part Γ_u, the boundary displacement is given: $u = \bar{u}$, on the remaining part Γ_t the surface traction $f = f(x)$ is prescribed. In the domain, the Green strain tensor ϵ is defined by the quadratic differential operator Λ:

$$\epsilon = \Lambda u = \frac{1}{2}[\nabla u + (\nabla u)^t + (\nabla u)^t(\nabla u)]. \tag{8}$$

The directional derivative of ϵ at u in the direction v is given by

$$\delta\epsilon(u; v) = \Lambda_t(u)v = \frac{1}{2}[\nabla v + (\nabla v)^t + (\nabla u)^t(\nabla v) + (\nabla v)^t(\nabla u)]. \tag{9}$$

Here the tangent mapping $\Lambda_t(u)$ is the Gâteaux derivative of $\epsilon(u) = \Lambda(u)u$. Its complementary mapping $\Lambda_n = \Lambda - \Lambda_t$ is given by

$$\Lambda_n(u)v = -\frac{1}{4}[(\nabla u)^t(\nabla v) + (\nabla v)^t(\nabla u)], \tag{10}$$

which plays a key role in finite deformation theory. By the Gauss–Green theorem, we have the virtual work principle:

$$\langle \sigma, \Lambda_t(u)v \rangle = \langle \Lambda_t^*(u)\sigma, v \rangle \quad \forall v \in \mathcal{U}, \tag{11}$$

where the adjoint operator Λ_t^* of Λ_t is defined by

$$\Lambda_t^*(u)\sigma = \begin{cases} -\nabla \cdot [(I + \nabla u)\sigma] & \text{in } \Omega \\ \mathbf{n} \cdot (I + \nabla u)\sigma & \text{on } \Gamma. \end{cases} \tag{12}$$

$\mathbf{n}$ is the unit vector normal to the boundary. The potential energy functions W and F in this system can be written as

$$W(\epsilon) = \begin{cases} \int_\Omega w(e)\, \mathrm{d}\Omega & \text{if } w \in \mathcal{L}^1(\Omega), \\ +\infty & \text{otherwise}, \end{cases} \tag{13}$$

$$F(u) = \begin{cases} -\int_\Omega bu\, \mathrm{d}\Omega - \int_{\Gamma_t} \bar{t}u\, \mathrm{d}\Gamma & \text{if } u = \bar{u} \\ +\infty & \text{otherwise.} \end{cases} \tag{14}$$

Where $w(\epsilon)$ is the stored energy function. The subdifferentiation of W gives the nonlinear elastic constitutive relation: $\sigma \in \partial W(\epsilon)$. For small strain deformation, $w(\epsilon)$ could be a quadratic function of the strain tensor: $w(\epsilon) = \frac{1}{2}\epsilon^t \mathbf{C}\epsilon$. In this case, this subdifferential inclusion degenerates to the linear elastic Hooke's law: $\sigma = \mathbf{C}\epsilon$. Meanwhile, the subdifferential of F gives the boundary condition:

$$-t \in \partial F(u) = \begin{cases} -b(\text{in } \Omega), -f(\text{on } \Gamma_t & \text{if } u = \bar{u} \\ 0 & \text{otherwise.} \end{cases}$$

So, in this example, the abstract governing equation (7) can be written as

$$-\nabla \cdot [(I + \nabla u)\sigma] = b \text{ in } \Omega$$

$$\mathbf{n} \cdot [(I + \nabla u)\sigma] = f \text{ on } \Gamma.$$

The traditional and the most commonly used finite-element method for solving nonlinear boundary-value problems is the displacement method, which is based on the primal variational problem (1). In this research, we study the complementary finite-element approach for nonsmooth systems, which is based on the dual variational problem.

For geometrical linear nonsmooth systems, the complementary energy J^*, i.e. the conjugate functional of $J(u, \Lambda u) = W(\Lambda u) + F(u)$ can be simply given by the Legendre-Fenchel transformation:

$$\begin{aligned} J^*(-\Lambda^*\sigma, \sigma) &= -\sup\sup\{\langle -\Lambda^*\sigma, u\rangle + \langle \sigma, \epsilon\rangle - J(u, \epsilon)\} \\ &= -W^*(\sigma) - F^*(-\Lambda^*\sigma). \end{aligned} \tag{15}$$

The complementary variational problem dual to the primal variational problem (1) then takes the following form:

$$(P^*) : \sup_{\sigma \in \mathcal{E}^*} J^*(-\Lambda^*\sigma, \sigma). \tag{16}$$

Since the conjugate functional J^* is concave and upper semi-continuous, if the general stress space $\mathcal{E}^*$ is a bounded, non-empty closed convex subset of a reflexive Banach space, the dual problem has at least one solution. If the primal functional $J(u, p) : \mathcal{U} \times \mathcal{E} \to \mathbf{R}$ is convex, then the complementary variational principle is equivalent to the primal variational problem (1) and

$$\inf P(u) = \sup P^*(\sigma).$$

In finite deformation systems, $\Lambda : \mathcal{U} \to \mathcal{E}$ is a quadratic nonlinear operator. The duality theory is established in [10]. By introducing the so-called complementary gap function defined by

$$G(\sigma, u) = \langle \sigma, -\Lambda_n u\rangle, \tag{17}$$

where Λ_n is the complementary operator of Λ : $\Lambda_n = \Lambda - \Lambda_t$, we found that the dual functional of $J(u, \Lambda u)$ should be

$$J^*(-\Lambda^*\sigma, \sigma) = -W^*(\sigma) - F^*(-\Lambda_t^*\sigma) - G(\sigma, u). \tag{18}$$

In the case of 3-dimensional nonlinear elastic systems, if the W and F are given by (13) and (14), respectively, their conjugate functions should be:

$$W^*(\sigma) = \sup_{\epsilon}\{\langle\sigma,\epsilon\rangle - W(\epsilon)\} = \begin{cases} \int_\Omega w^*(\sigma)\,\mathrm{d}\Omega & \text{if } w^*(\sigma) \in \mathcal{L}^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \tag{19}$$

$$F^*(-\Lambda_t^*\sigma) = \begin{cases} \int_{\Gamma_u} \mathbf{n}\cdot(I+\nabla u)\sigma\bar{u}\,\mathrm{d}\Gamma & \text{if } \Lambda_t^*\sigma = \bar{t} \\ +\infty & \text{otherwise.} \end{cases} \tag{20}$$

For the finite deformation operator Λ defined by eqn. (8), the complementary gap function should be

$$G(\sigma,u) = \langle\sigma, -\Lambda_n(u)u\rangle = \int_\Omega \frac{1}{2}\mathrm{tr}[(\nabla u)^t\sigma(\nabla u)]\,\mathrm{d}\Omega. \tag{21}$$

We proved that the stationary condition of J^* gives the same Euler-Lagrange inclusion as J. Then the dual approach of the primal problem can be proposed by

PROBLEM 2. *Find σ, u such that*

$$(P^*) : J^*(-\Lambda^*(u)\sigma,\sigma) = \sup_{(\tau,v)\in\mathcal{E}^*\times\mathcal{U}} J^*(-\Lambda^*(v)\tau,\tau) \tag{22}$$

THEOREM 1 (Gao and Strang [10]). *Suppose that (u,σ) is a critical point of the dual functional J^*, i.e. $\delta J^*(-\Lambda^*(u)\sigma,\sigma) = 0$. If the complementary gap function $G(\sigma,v) \geqslant 0\,\forall v \in \mathcal{U}$, then the dual variational problem has at least one solution (σ,u) and*

$$\inf J(v,\Lambda v) = \sup J^*(-\Lambda^*(v)\tau,\tau). \tag{23}$$

If the gap function is strictly positive, the dual problem has a unique solution.

This theorem shows that the complementary gap function provides a global extremum criterion for geometrical nonlinear variational problems. Based on this general theory, a series of complementary variational principles for finite deformation nonsmooth mechanics have been established (see [11–16]). Theoretical analysis shows that for finite-deformation systems governed by the linear physical laws (i.e. large displacements and small-strain systems), such as thin-walled elastic structures, the primal problem is a non-linear variational problem. But the dual problem is a quadratic optimization problem. Based on this complementary variational theory, we can suggest an effective algorithm for solving the geometrical nonlinear mixed boundary-value problems.

3. Finite element approach and algorithm

In this section, we present the finite-element approach for the dual variational problem

$$\sup_{(\tau,v)\in\mathcal{E}^*\times\mathcal{U}} \{J^*(-\Lambda^*(v)\tau,\tau) = -W^*(\tau) - F^*(-\Lambda_t^*\tau) - G(\tau,v)\},$$

Here we assume that W^* is strictly convex. For mixed boundary-value problems in finite-deformation theory, the functional $F^*(\tau,v)$ is given in (20). If we let

$$P^*(\sigma,v) = -W^*(\sigma) + \langle\Lambda_t^*\sigma,\bar{u}\rangle - G(\sigma,v),$$

then the dual problem (P^*) can be written as

$$\sup\{P^*(\tau, v)|\Lambda_t^*(v)\tau = \bar{t}\}. \tag{24}$$

This is an infinite dimensional nonsmooth optimization with equality constraint.

Suppose the domain Ω can be discretized by finite elements such that $\Omega = \cup_h \Omega^h$. In each element Ω^h, the equilibrium constraint in the nonsmooth optimization problem (24) can be relaxed by the following weak form:

$$\int_{\Omega^h} [(\Lambda_t v)\tau - vb] \, \mathrm{d}\Omega^h - \int_{\partial\Omega^h} v\Lambda_t^* \tau \, \mathrm{d}\partial\Omega^h = \int_{\Gamma_t^h} f v \, \mathrm{d}\Gamma^h \quad \forall v \in \mathcal{U}, \tag{25}$$

in which $\partial\Omega^h$ is the boundary of the element Ω^h not belonging to $\partial\Omega$. By introducing the suitable independent interpolation for τ and v in each element:

$$\tau^h(x) = \mathbf{F}_\tau^e(x)\mathbf{q}^h, \qquad v^h(x) = \mathbf{F}_v^e(x)\mathbf{p}^h \quad \forall x \in \Omega^h, \tag{26}$$

where $\mathbf{F}_\tau^e(x)$ and $\mathbf{F}_v^e(x)$ are interpolation matrices expressing local values of τ^h and v^h in terms of the element parameters $\mathbf{q}^h$ and $\mathbf{p}^h$, respectively, the complementary finite-element formulation for the dual problem (P^*) can be given as:

$$(P_h^*): \begin{array}{l} \max_{\mathbf{q}\in\mathbf{R}^n} \max_{\mathbf{p}\in\mathbf{R}^m} P^*(\mathbf{q}, \mathbf{p}) \\ \text{s.t. } \mathbf{B}(\mathbf{p})\mathbf{q} - \mathbf{Q} = 0, \end{array} \tag{27}$$

where $\mathbf{B}(\mathbf{p})$ is a $m \times n$ equilibrium matrix, which is given by

$$\begin{aligned} \mathbf{B}(\mathbf{p}) = & \sum_h \Big\{ \int_{\Omega^h} (\Lambda_t(\mathbf{F}_v^e(x)\mathbf{p}^h)\mathbf{F}_v^e(x))^t \mathbf{F}_\tau^e(x) \, \mathrm{d}\Omega \\ & - \int_{\partial\Omega^h} (\mathbf{F}_v^e(x))^t \mathbf{n}(I + \nabla \mathbf{F}_v^e(x)\mathbf{p}^h)\mathbf{F}_\tau^e(x) d(\partial\Omega^h) \Big\}, \end{aligned}$$

and $\mathbf{Q} \in \mathbf{R}^m$ is the nodal external force vector:

$$\mathbf{Q} = \sum_e \left\{ \int_{\Omega^h} (\mathbf{F}_v^e(x))^t b \, \mathrm{d}\Omega^h + \int_{\Gamma_t^h} (\mathbf{F}_v^e(x))^t t d\Gamma^h \right\}.$$

THEOREM 2. *For any given finite-element discretization of problem (P^*), if the complementary gap function*

$$G(\tau(\mathbf{q}), v(\mathbf{p})) \geqslant 0 \quad \forall \mathbf{q} \in \mathbf{R}^n, \mathbf{p} \in \mathbf{R}^m, \tag{28}$$

and the following rank condition holds,

$$\mathit{rank}\, \mathbf{B} = m < n, \tag{29}$$

then the discrete dual problem (P_h^*) has at least one solution $(\mathbf{q}^h, \mathbf{p}^h)$. It has a unique solution if the gap function is strictly positive.

Proof. For any given finite-element interpolation (26), if the gap function $G(\tau(\mathbf{q}), v(\mathbf{p}))$ is positive, then $P^*(\mathbf{q}, \mathbf{p}) : \mathbf{R}^m \to \mathbf{R}$ is quadratic and concave for any given $\mathbf{q} \in \mathbf{R}^n$ and

$$\max_{\mathbf{p}\in\mathbf{R}^m} P^*(\mathbf{q}, \mathbf{p}) \quad \forall \mathbf{q} \in \mathbf{R}^n$$

has at least one solution $\mathbf{p}^h$. Since the general solution of the equilibrium equation $\mathbf{B}(\mathbf{p}^h)\mathbf{q} - \mathbf{Q} = 0$ is a linear manifold, we have

$$\mathbf{q} = \mathbf{q}_o + \mathcal{N}(\mathbf{B}(\mathbf{p}^h)), \tag{30}$$

where $\mathbf{q}_o$ is a particular inhomogeneous solution:

$$\mathbf{B}\mathbf{q}_o - \mathbf{Q} = 0. \tag{31}$$

$\mathcal{N}(\mathbf{B}) \subset \mathbf{R}^n$ denotes the null space of linear operator $\mathbf{B}$. If the rank condition (29) is true, then the nullity of $\mathbf{B}$ satisfies:

$$\eta(\mathbf{B}) = \dim \mathcal{N}(\mathbf{B}(\mathbf{p}^h)) = n - m = r > 0, \tag{32}$$

and the particular inhomogeneous solution $\mathbf{q}_o$ can be given by

$$\mathbf{q}_o = \mathbf{B}^+(\mathbf{p}^h)\mathbf{Q}, \tag{33}$$

where $\mathbf{B}^+$ is so-called Moore-Penrose inverse of matrix $\mathbf{B}$, which satisfies following conditions:

$$\begin{aligned} &\mathbf{B}^+\mathbf{B}\mathbf{B}^+ = \mathbf{B}^+ \qquad &&\mathbf{B}\mathbf{B}^+\mathbf{B} = \mathbf{B} \\ &(\mathbf{B}^+\mathbf{B})^t = \mathbf{B}^+\mathbf{B}, \qquad &&(\mathbf{B}\mathbf{B}^+)^t = \mathbf{B}\mathbf{B}^+. \end{aligned} \tag{34}$$

Since the rank of $\mathbf{B}$ is equal to the number of its rows, $\mathbf{B}^+$ can be constructed as

$$\mathbf{B}^+ = \mathbf{B}^t(\mathbf{B}\mathbf{B}^t)^{-1}. \tag{35}$$

For any given $\mathbf{q}_r \in \mathbf{R}^r$, the null space $\mathcal{N}(\mathbf{B})$ can be constructed by

$$\mathcal{N}(\mathbf{B}) = \{\mathbf{q}_n \in \mathbf{R}^n | \mathbf{q}_n = \mathbf{N}\mathbf{q}_r \quad \forall \mathbf{q}_r \in \mathbf{R}^r\}, \tag{36}$$

in which $\mathbf{N} = \mathbf{P}^*\mathbf{D}^*$, $\mathbf{P}^*$ is a n by n complementary projector of the linear operator $\mathbf{B}$:

$$\mathbf{P}^* = \mathbf{I} - \mathbf{B}^+\mathbf{B} = \mathbf{I} - \mathbf{P}, \tag{37}$$

where $\mathbf{P}$ is the project matrix of $\mathbf{B}$, and $\mathbf{D}^* \in \mathbf{R}^n \times \mathbf{R}^r$ is a matrix such that the row vectors of $\mathbf{P}^*\mathbf{D}^*$ is a base of row vectors of $\mathbf{P}^*$. By taking the property of $\mathbf{B}^+$ into account, it is obvious that

$$\mathbf{B}\mathbf{N}\mathbf{q}_r = \mathbf{B}\mathbf{P}^*\mathbf{D}^*\mathbf{q}_r = \{\mathbf{B} - \mathbf{B}\mathbf{B}^+\mathbf{B}\}\mathbf{D}^*\mathbf{q}_r \equiv 0 \quad \forall \mathbf{q}_r \in \mathbf{R}^r,$$

which means that $\mathbf{N}\mathbf{q}_r \in \mathcal{N}(\mathbf{B})$. Therefore, the general solution of the equation $\mathbf{B}(\mathbf{p}^h)\mathbf{q} - \mathbf{Q} = 0$ can be written as

$$\mathbf{q} = \mathbf{q}_o + \mathbf{N}\mathbf{q}_r \quad \forall \mathbf{q}_r \in \mathbf{R}^r. \tag{38}$$

Substituting this general solution into the discrete dual problem (P_h^*), we may reduce the maximizing of P^* for $\mathbf{q}$ to

$$\max_{\mathbf{q}_r \in \mathbf{R}^r} P^*(\mathbf{q}(\mathbf{q}_r), \mathbf{p}^h). \tag{39}$$

This is a nonsmooth optimization problem with only $r = n - m$ degrees-of-freedom. Since $P^* : \mathbf{R}^n \to \mathbf{R}$ is strictly concave on $\mathbf{q}$ for any given $\mathbf{p} \in \mathbf{R}^m$, this problem has only one solution $\mathbf{q}_r^h$. If the gap function is strictly positive, $P^* : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ is strictly concave on both $\mathbf{q} \in \mathbf{R}^n$ and $\mathbf{p} \in \mathbf{R}^m$. If the rank condition (29) is true, the dual problem (P_h^*) possesses only one solution $(\mathbf{q}^h, \mathbf{p}^h)$. Q.E.D.

Based on this theorem, we can suggest the following algorithm for solving the coupled convex algorithm (27):

(1) For a given $\mathbf{p}^k \in \mathbf{R}^m$, solve the discrete equilibrium equation

$$\mathbf{B}(\mathbf{p}^k)\mathbf{q} = \mathbf{Q}$$

by the method given in the proof of Theorem 2. We have

$$\mathbf{q} = \mathbf{q}_o^k + \mathbf{N}(\mathbf{p}^k)\mathbf{q}_r. \tag{40}$$

(2) Solve the nonlinear (or nonsmooth) programming problem

$$\max_{\mathbf{q}_r \in \mathbf{R}^r} P^*(\mathbf{q}_o^k + \mathbf{N}\mathbf{q}_r, \mathbf{p}^k) \tag{41}$$

for $\mathbf{q}_r^k$.

(3) Let $\mathbf{q}^k = \mathbf{q}_o^k + \mathbf{N}(\mathbf{p}^k)\mathbf{q}_r^k$, solve the problem

$$\max_{\mathbf{p} \in \mathbf{R}^m} P^*(\mathbf{q}^k, \mathbf{p}) \tag{42}$$

obtaining $\mathbf{p}^{k+1}$.

(4) If $\|\mathbf{B}(\mathbf{p}^{k+1})\mathbf{q}^k - \mathbf{Q}\| \leqslant \omega$ ($\omega > 0$ is a previously given constant), stop. Otherwise, let $k = k + 1$, go back to step (1).

By the theorem proved above, if the gap function G possess a right sign and the rank condition (29) is true, then the sequence $\{\mathbf{p}^k, \mathbf{q}^k\}$ will converge to the solution of the primal problem.

4. Applications

Consider a rigid perfect-plastic material occupying a volume Ω in three-dimensional Euclidean space $\mathbf{R}^3$ with boundary $\partial\Omega = \Gamma_u \cup \Gamma_t$. Let u denote the velocity of the particle $x = \{x^\alpha\} \in \Omega$. For given external loading system b (in Ω), f (on Γ_t), the governing equations for plastic flow are given as follows:

$$\epsilon = \Lambda u \quad \text{in } \Omega, \qquad u = 0 \text{ on } \Gamma_u, \tag{43}$$

$$\sigma \in \partial\Phi_K(\epsilon) \quad \text{or} \quad \epsilon \in \partial\Psi_K(\sigma) \quad \text{in } \Omega, \tag{44}$$

$$\Lambda_t^*(u)\sigma = b \quad \text{in } \Omega \quad \Lambda_t^*(u)\sigma = f \quad \text{on } \Gamma_t. \tag{45}$$

For small displacement problems, Λ is a gradient-like operator: $\Lambda = \frac{1}{2}(\nabla + \nabla^t)$ and its adjoint $\Lambda^* = \Lambda_t^*$ should be the divergence operator

$$\Lambda^*\sigma = \begin{cases} -\nabla \cdot \sigma & \text{in } \Omega, \\ n \cdot \sigma & \text{on } \Gamma. \end{cases} \tag{46}$$

For large deformation problems, Λ is given by (10) (see Gao and Strang [11]). $\Phi_K(\epsilon)$ is the support function of the convex set K:

$$\Phi_K(\epsilon) = \sup_{\sigma \in K} \{\langle \sigma, \epsilon \rangle\}. \tag{47}$$

The convex set K is defined as

$$K = \{\sigma \in L^q(\Omega, \mathbf{R}^3 \times \mathbf{R}^3) | f_i(\sigma) \leqslant 0, \quad i = 1, 2, \ldots, n\}, \tag{48}$$

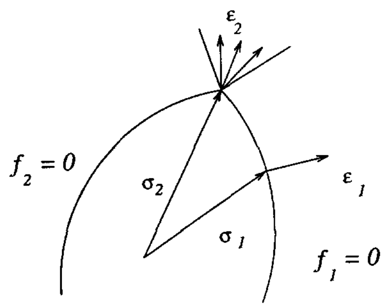

Fig. 1. Nonsmooth plastic constitutive relation.

where $f_i(\sigma)$ is the so-called plastic yield function, which is a convex function of stress. For a von-Mises material, $f_i(\sigma) = \sqrt{\sigma^d_{ij}\sigma^d_{ij}} - \sigma_b$. σ^d is the stress deviator, $\sigma^d = \sigma - \mathrm{tr}\,\sigma$. Ψ_K is the conjugate function of Φ_K given by the following Legendre-Fenchel transformation:

$$\Psi_K(\sigma) = \sup_{\epsilon}\{\langle\epsilon,\sigma\rangle - \Phi_K(\epsilon)\} = \begin{cases} 0 & \text{if } \sigma \in K \\ +\infty & \text{if } \sigma \notin K. \end{cases} \tag{49}$$

which is called the indicator function of the convex set K. This is a nonsmooth system. The constitutive relation between the strain and stress tensor

$$\epsilon \in \partial\Phi_K(\sigma) = \begin{cases} \sum_i \mu_i \dfrac{\partial f_i(\sigma)}{\partial\sigma} & \text{if } f_i(\sigma) = 0, \mu_i \geqslant 0, \\ \{0\} & \text{if } f_i(\sigma) < 0, \\ \emptyset & \text{if } f_i(\sigma) > 0. \end{cases} \tag{50}$$

is a point-to-set mapping, see Fig. 1. At $\sigma = \sigma_1$, $\partial\Psi_K(\sigma)$ has the only one element $\epsilon_1 = \mu\partial f_1(\sigma)/\partial\sigma$, $\mu \geqslant 0$ is the plastic-flow factor. But at $\sigma = \sigma_2$, the intersection of f_1 with f_2, $\partial\Psi_K$ is a convex cone. Any vector in this cone is the possible strain variable associated with the given stress field σ_2.

Actually, by the definition of the subdifferential $\partial\Psi_K$:

$$\partial\Psi_K(\sigma) = \{\epsilon \in \mathcal{E} | \Psi_K(\tau) - \Psi_K(\sigma) \geqslant \langle\epsilon, \tau - \sigma\rangle, \quad \forall\tau \in \mathcal{E}^*\}.$$

It is obvious that if both $\tau, \sigma \in K$, this subdifferential set $\partial\Psi_K$ is equivalent to

$$\langle\epsilon(\sigma), \sigma - \tau\rangle \geqslant 0 \quad \forall\tau, \sigma \in K.$$

This is the well-known maximum work theorem.

Let

$$W(\epsilon) = \int_\Omega \Phi_K(\epsilon)\,\mathrm{d}\Omega, \quad F(u) = \begin{cases} -\int_\Omega bu\,\mathrm{d}\Omega - \int_{\Gamma_t} tu\,\mathrm{d}\Gamma \\ +\infty, \end{cases} \tag{51}$$

then the mixed-boundary value problem (43–45) can be written as the following optimization problem:

$$\inf_{u \in H^1_0(\Omega,\mathbf{R}^3)} \{W(\Lambda u) + F(u)\}. \tag{52}$$

This is a nonsmooth variational problem. For plastic limit analysis, we take the linear deformation operator Λ. (The nonlinear plastic limit analysis is studied in [11].) Further, the external load is suppose to be proportional to a positive scalar $\nu > 0$, i.e. $\mathbf{t} = (b, f) = \nu(\bar{b}, \bar{f}) = \nu\bar{\mathbf{t}}$. Over the kinematically admissible space defined by

$$\mathcal{U}_a := \left\{ u \in \mathcal{L}^1(\Omega) | u = 0 \text{ on } \Gamma_u, \int_\Omega \bar{f} u \, \mathrm{d}\Omega + \int_{\Gamma_t} \bar{t} u \, \mathrm{d}\Gamma = 1 \right\}$$

the collapse load factor ν_c for limit analysis can be given as

$$(P) : \nu_c = \inf_{u \in \mathcal{U}_a} W(\Lambda u). \tag{53}$$

The Ritz finite-element method for this primal problem will provide an upper-bound approach for the collapse load factor ν_c.

The complementary energy in this problem is (see [19])

$$W^*(\sigma) = \int_\Omega \Phi_K(\sigma) \, \mathrm{d}\Omega.$$

Let $\mathcal{S}_a$ be the statically admissible space defined as

$$\mathcal{S}_a := \{ \sigma \in \mathcal{E}^* | \Lambda^* \sigma = \nu^- \bar{t} \},$$

in which $\nu^- > 0$ is the statically admissible load factor determined by $\sigma \in \mathcal{S}_a$. Then the dual problem for limit analysis is given in [19]:

$$(P^*) : \nu_c = \sup_{\tau \in \mathcal{S}_a} \{ \nu^-(\tau) - W^*(\tau) \}. \tag{54}$$

This dual problem will provide the lower bound approach for the collapse load factor ν_c. If the statically admissible stress field τ is in the yield set K, then (54) can be written as

$$\nu_c = \sup\{ \nu^-(\tau) | \Lambda^* \tau = \nu^- \bar{t}, \quad f_i(\tau) \leqslant 0 \}.$$

This is the classical lower-bound theorem.

Assuming suitable independent interpolation rules for both stress and velocity fields in each element Ω^h, we have:

$$\tau^h(x) = \nu^- \mathbf{F}^e_\tau(x) \mathbf{q}^h, \qquad v^h(x) = \mathbf{F}^e_v(x) \mathbf{p}^h, \tag{55}$$

the discrete form for dual problem (54) may be represented as

$$(P^*_h) : \begin{array}{l} \nu^h_c = \max_{(\nu^-, \mathbf{q}) \in \mathbf{R}^+ \times \mathbf{R}^n} \{ \nu^- - W^*(\nu^-, \mathbf{q}) \}, \\ \text{s.t. } \mathbf{B}\mathbf{q} - \mathbf{Q} = \mathbf{0}, \end{array} \tag{56}$$

in which

$$\mathbf{B} = \sum_h \left\{ \int_{\Omega^h} (\nabla \mathbf{F}^e_v(x))^t \mathbf{F}^e_\tau(x) \, \mathrm{d}\Omega - \int_{\partial\Gamma^h} (\mathbf{F}^e_v(x))^t \mathbf{n} \mathbf{F}^e_\tau(x) d(\partial\Omega^h) \right\}, \tag{57}$$

$$\mathbf{Q} = \sum_e \left\{ \int_{\Omega^h} (\mathbf{F}^e_v(x))^t b \, \mathrm{d}\Omega^h + \int_{\Gamma^h_t} (\mathbf{F}^e_v(x))^t t d\Gamma^h \right\}, \tag{58}$$

$$W^*(\nu^-, \mathbf{q}) = \sum_h \int_{\Gamma^h} \Psi_K(\tau(\nu^-, \mathbf{q}))\, \mathrm{d}\Omega^h. \tag{59}$$

If the yield set K is convex, and the rank condition (29) holds, by the algorithm suggested above, this dual problem has a unique solution and, for any given finite element discretization Ω^h of Ω, we have a lower-bound approach for the collapse factor

$$\nu_c \geqslant \nu_c^h, \quad \nu_c = \lim_{h \to 0} \nu_c^h$$

In order to solve the nonsmooth optimization

$$\max_{\mathbf{q}_r \in \mathbf{R}^r} \{\nu^- - W^*(\nu^-, \mathbf{q}_o + \mathbf{N}\mathbf{q}_r)\},$$

an augmented Lagrange method (see [20]) can be used here. According to the property of W^*, we let

$$W^*_{pd}(\nu^-, \mathbf{q}, \lambda, \alpha) = \sum_h \int_{\Omega^h} \frac{\alpha}{2} \left\{ \left[\lambda + \frac{1}{\alpha} f(\sigma(\nu^- \mathbf{q})) \right]^2 \phi(\psi^h) - \lambda^2 \right\} \mathrm{d}\Omega^h, \tag{60}$$

in which $\alpha > 0$ is the penalty factor, $\lambda \geqslant 0$ is the dual variable of the yield function $f(\sigma)$, ϕ is the jump function:

$$\phi(\psi^h) = \begin{cases} 1 & \text{if } \psi^h > 0, \\ 0 & \text{if } \psi^h \leqslant 0. \end{cases}$$

ψ^h is a domain-dividing function: $\psi^h = \lambda + (1/\alpha) f(\sigma)$ in Ω^h (see [20]). It is easy to prove that for any given $\mathbf{q} \in \mathbf{R}^n$, and $\nu^- > 0$, we have

$$W^*(\nu^-, \mathbf{q}) = \max_{\alpha > 0, \lambda \geqslant 0} W^*_{pd}(\nu^-, \mathbf{q}, \lambda, \alpha). \tag{61}$$

Substituting this into (56) and letting

$$\nu^h_{pd}(\nu^-, \mathbf{q}, \lambda, \alpha) = \nu^- - W^*_{pd}(\nu^-, \mathbf{q}, \lambda, \alpha),$$

we may give the penalty-duality finite-element approach for nonsmooth plastic limit analysis as follows:

$$\max_{\nu^- > 0} \max_{\mathbf{q}_r \in \mathbf{R}^r} \min_{\lambda \geqslant 0, \alpha > 0} \nu^h_{pd}(\nu^-, \mathbf{q}_o + \mathbf{N}\mathbf{q}_r, \lambda, \alpha). \tag{62}$$

In [20], the penalty-duality iterative algorithm for this nonsmooth optimization can be suggested as follows:

Given a penalty-duality factor $\alpha_k > 0$, $\lambda_k \geqslant 0$, determine ν_k^-, $\mathbf{q}_r^k$ by

$$\max_{\nu^-, \mathbf{q}_r} \nu^h_{pd}(\nu^-, \mathbf{q}_o + \mathbf{N}\mathbf{q}_r, \lambda_k, \alpha_k). \tag{63}$$

Then modify the penalty-duality factor by

$$\lambda_{k+1} = \left\{ \lambda_k + \frac{1}{\alpha_k} f(\sigma(\nu_k^-, \mathbf{q}_k)) \right\} \phi(\psi^h) \quad \text{in } \Omega_h, \tag{64}$$

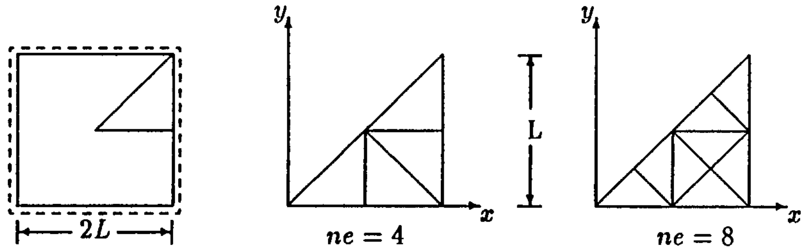

Fig. 2. Simply support square plate and finite-element meshes.

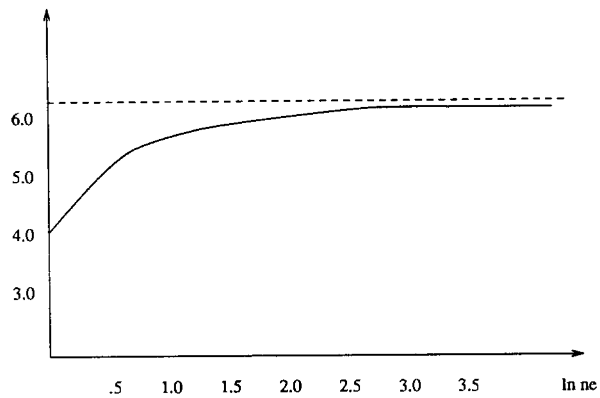

Fig. 3. Convergence tests of simply supported square plate.

$$\alpha_{k+1} = \begin{cases} \alpha_k & \text{if } |f(\sigma(\nu_k^-, \mathbf{q}_k))| \leqslant \theta |f(\sigma(\nu_{k-1}^-, \mathbf{q}_{k-1}))| \\ \gamma\alpha_k & \text{otherwise,} \end{cases} \tag{65}$$

where $\theta \in [0.1, 0.4]$ and $\gamma \in [0.1, 0.25]$ are control parameters.

Let us consider the limit analysis for a rigid perfectly plastic plate. The finite-element discretization is based on triangular elements with linear variation for the transversal displacements and constant for the bending moments. For small deformation theory, using the Von-Mises yield function:

$$f(m_{ij}) = \sqrt{m_x^2 + m_y^2 - m_x m_y + 3m_{xy}^2} - m_b, \tag{66}$$

where, with obvious meaning of the symbols, m_x, m_y and m_{xy} are the bending moments and m_b is the flexural strength of the plate. For the simply-supported circular plate of uniform thickness subjected to a uniform distributed load, with only 41 elements meshes, the collapse load is given as $\nu_c^h = 6.50 m_b/a^2$. The exact collapse factor is known as $\nu_c = 6.51 m_b/a^2$; the relative error $(\nu_c^h - \nu_c)/\nu_c$ is only 0.015.

The analysis of a square plate of uniform thickness is performed by using a different series of meshes shown in Fig. 2, with the aim of performing a numerical test of convergence (see Fig. 3).

Table 1 shows a comparison of the numerical results obtained by the proposed method with the existing bounds and approximate values of the collapse load. The results show that the complementary finite-element method provides a very good lower-bound approach for this nonsmooth system.

Table 1. Collapse load of simply supported square plate.

Belytschko–Hodge [21]		Ranawerra–Leckie [22]		Gao	Casciaro–Cascini	Christiansen
ν^+	ν^-	ν^+	ν^-	ν_c^h	[23]	and Larsen [24]
6.635	6.216	6.265	5.97	6.249	6.258	6.255

5. Concluding remarks

(1) It is known that in the large-scale structural finite-element analysis, nonlinear iteration is time-consuming. Assume a given finite-element discretization such that the nodal displacement vector $\mathbf{p} \in \mathbf{R}^m$, and nodal stress vector $\mathbf{q} \in \mathbf{R}^n$. Then by the traditional displacement method, in each iteration, we have to solve the nonlinear problem min $P(v(\mathbf{p}))$ with m degrees of freedom. However, by using the algorithm suggested in this paper, we can use the equilibrium constraint $\mathbf{Bq} = \mathbf{Q}$ in problem (27) to reduce the degrees of freedom in the nonlinear programming problem (41) to $r = n - m$ only. Since r is much less than m, this algorithm can save much more computing time than the Ritz method.

Let us take the plane plastic flow as an example. Suppose the two-dimensional domain is divided by finite elements with a total N nodes. At each node, the velocity vector $\mathbf{p}^h$ has two degrees-of-freedom, and the nodal stress vector $\mathbf{q}^h$ has 3. So the total degree-of-freedom for the Ritz method will be dim $\mathbf{p} = 2N$. But, by the method we proposed, the degree-of-freedom for nonlinear programming is only $r = \dim \mathbf{q}_r = 3N - 2N = N$. This fact shows that this complementary finite-element method can reduce the degrees-of-freedom in large-scalar nonlinear programming to a great extent.

(2) In large displacement but small-strain deformation systems, the complementary energy W^* is a quadratic function of the generalized stress, and the gap function $G(\sigma, v)$ is a quadratic function of the displacement. In this case, the primal problem is still a weak nonlinear variational problem due to the finite deformation operator Λ. But, the dual problem $\sup J^*(-\Lambda^*(v)\sigma, \sigma)$ is a coupled quadratic variational problem. So this proposed algorithm provides a coupled quadratic approach for solving the nonlinear boundary-value problems.

(3) For nonsmooth systems, the functional $W(\epsilon)$ and $F(v)$ could be nonconvex. But by the Legendre-Fenchel transformation, the dual functionals $W^*(\sigma)$ and $F^*(t)$ are always convex. Then we can use the so-called "bundle method" (see, for example, C. Lemarchel, J.J. Strodiot and A. Bihain [25], K. Kiwiel [26], J.J. Strodiot and V.H. Nguyen [27]) to solve the nonsmooth programming problem.

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